Substitution of Eq. (15) into Eq. (12) provides the desired boundary condition.

As a special case, consider the unidirectional, surface-catalyzed reaction

$$n\mathfrak{M}_1 \xrightarrow{k_f} \mathfrak{M}_2 \tag{16}$$

so that $K' = 1, N' = 2$; the general relation yields

$$b_1 = \frac{k_f n(\gamma_0 \kappa_1 c_1)^n}{(1 + \kappa_1 c_1 + \kappa_2 c_2)^n} \tag{17}$$

so that for highly dilute reactants and products the frequently employed relation $b_1 \sim c_1{}^n$ is recovered. However, if $\kappa_1 c_1$ is large compared with $\kappa_2 c_2$ and with unity, then Eq. (17) indicates that b_1 is independent of c_1, i.e., the reaction is of zero order.[11] Especially interesting is the limit $\kappa_2 c_2 \gg 1$, $\kappa_2 c_2 \gg \kappa_1 c_1$, which is more likely to occur downstream, where the reaction has proceeded in the forward direction to a significant extent; in this case $b_i \sim (c_1/c_2)^n$, thus exhibiting retardation of the reaction rate by the reaction product. Clearly, the effective order of the reaction may change with the streamwise coordinate in a given boundary layer.

It is also interesting to note that for a more complicated unidirectional reaction of the form

$$\mathfrak{M}_1 + \mathfrak{M}_2 \xrightarrow{k_f} \mathfrak{M}_3 + \mathfrak{M}_4 \tag{18}$$

the general relation for the flux of species 1, e.g., yields

$$b_1 = \frac{\gamma_0{}^2 k_f \kappa_1 \kappa_2 c_1 c_2}{(1 + \kappa_1 c_1 + \kappa_2 c_2 + \kappa_3 c_3 + \kappa_4 c_4)^2} \tag{19}$$

Clearly, depending on the magnitudes of each product $\kappa_1 c_1$ compared to unity, a range of reaction orders from plus to minus unity can occur.

In conclusion, it is noted that the foregoing considerations apply to turbulent as well as to laminar boundary layers.

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Generalized Matrix Force and Displacement Methods of Linear Structural Analysis

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Nomenclature

Force method

P	=	column matrix‡ of all generalized forces
P_0, P_1	=	column matrices of known and unknown generalized forces, respectively
b_g, b_{0g}, b_{1g}	=	rectangular matrices of stress resultants (internal forces) in element g due to unit values of components of P, P_0, P_1, respectively
b, b_0, b_1	=	rectangular matrices of stress resultants in all elements due to unit values of components of P, P_0, P_1, respectively
S_g	=	column matrix of stress resultants in element g due to P
S, S_0, S_1	=	column matrices of stress resultants in all elements due to P, P_0, P_1, respectively
w_g	=	column matrix of strain resultants (over-all internal deformations) of element g, corresponding to S_g
w, w_0, w_1	=	column matrices of strain resultants of all elements, corresponding to S, S_0, S_1
p, p_0, p_1	=	column matrices of all displacements in the directions of forces P, P_1, P_0, respectively
f_g	=	square matrix describing flexibility of element g
f	=	square matrix of flexibilities of all elements (unassembled)

Displacement method

r	=	column matrix of all displacements
r_0, r_1	=	column matrices of known and unknown displacements, respectively
a_g, a_{0g}, a_{1g}	=	rectangular matrices of strain resultants (displacements of extremities) of element g due to unit values of components of r, r_0, r_1, respectively
a, a_0, a_1	=	rectangular matrices of the strain resultants of all elements due to unit values of components of r, r_0, r_1, respectively
v_g	=	column matrix of strain resultants of element g due to r
v, v_0, v_1	=	column matrices of strain resultants of all elements due to r, r_0, r_1, respectively
T_g	=	column matrix of stress resultants (forces at extremities) in element g, corresponding to v_g
T, T_0, T_1	=	column matrices of stress resultants in all elements, corresponding to v, v_0, v_1
R, R_0, R_1	=	column matrices of all forces in the directions of displacements r, r_1, r_0, respectively
k_g	=	square matrix describing stiffness of element g
k	=	square matrix of stiffnesses of all elements (unassembled)

Introduction

THAT there are two related possible approaches to the analysis of structures has long been realized. Ostenfeld,[1] however, first fully outlined the principles that underlie the duality between the force (or flexibility) method and the displacement (or stiffness) method. Argyris[2] expressed the relationship between the two methods in matrix notation and put forward procedures for the solution of structures under either applied forces or imposed displacements.

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‡ Column and diagonal matrices, when shown in expanded form, are denoted, respectively, by { } and ⌜ ⌟.

The generalized matrix equations of the force and displacement methods for a structure under any combination of applied forces and imposed displacements are given now. Their derivation is based upon topological and geometrical reasoning. Such an approach provides an appreciation of the contribution of each element to the complete structure.

Force Method

Basic theory

In the force method, a system of self-equilibrating forces is considered as a single generalized force, and the associated displacements are the relative deformations within the system. The behavior of a structure is then fully defined when all of the generalized forces are known. The initial conditions provide knowledge of either each generalized force or its corresponding displacement.

The forces P on the entire structure produce stress resultants S in all of the elements. Then, for the element g,

$$S_g = b_g P \tag{1}$$

where b_g is obtained by considering one generalized force at a time. The matrix b_g, expressing equilibrium, thus describes a topological and geometrical relationship, i.e., the configuration of the structure. The strain resultants w_g are given by

$$w_g = f_g S_g \tag{2}$$

Assuming that only element g has flexibility and that all of the others have none (they are infinitely rigid), it will be the only element under strain resultants. The contribution of w_g to all of the displacements of the structure is defined as p_g. Then, by the same topological reasoning used in the derivation of Eq. (1),

$$p_g = b_g' w_g \tag{3}$$

This equation is an expression of the principle of virtual forces.

By considering one-by-one the flexibilities of all of the elements and their contributions to the displacements of the whole structure, the resulting displacements Σp_g then become equal to the displacements p; thus

$$p = \Sigma p_g \tag{4}$$

Then, by substitution,

$$p = \Sigma b_g' w_g = \Sigma(b_g' f_g S_g) = [\Sigma(b_g' f_g b_g)]P \tag{5}$$

Defining

$$D = \Sigma b_g' f_g b_g \tag{6}$$

then

$$p = DP \tag{7}$$

where D is the flexibility matrix of the structure and relates all of the generalized forces considered to the displacements.

Equation (7) therefore provides a complete solution for the structure acted upon by forces and/or deformations.

Method of solution

Partitioning the matrices of Eq. (7) gives

$$\begin{bmatrix} p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \end{bmatrix}$$

whence, with D_{01} equal to D_{10}',

$$p_1 = D_{00}P_0 + D_{01}P_1 \tag{8}$$

$$p_0 = D_{10}P_0 + D_{11}P_1 \tag{9}$$

Equation (9) gives the unknown forces

$$P_1 = D_{11}^{-1}(p_0 - D_{10}P_0) \tag{10}$$

Displacement Method

Basic theory

In the displacement method, a system of compatible displacements is considered as a single generalized displacement, and the associated forces are those within the system. The behavior of a structure is then fully defined when all of the generalized displacements are known. The initial conditions provide knowledge of either each generalized displacement or its corresponding force.

The displacements r of the entire structure produce strain resultants v in all of the elements. Then, for the element g,

$$v_g = a_g r \tag{1}$$

where a_g is obtained by considering one displacement at a time. The matrix a_g, expressing compatibility, thus describes a topological and geometrical relationship, i.e., the configuration of the structure. The stress resultants T_g are given by

$$T_g = k_g v_g \tag{2}$$

Assuming that only the element g has stiffness and that all of the others have none (they effectively do not exist), it will be the only element carrying forces. The contribution of T_g to all of the forces on the structure is defined as R_g. Then, by the same topological reasoning used in the derivation of Eq. (1),

$$R_g = a_g' T_g \tag{3}$$

This equation is an expression of the principle of virtual displacements.

By considering one-by-one the stiffnesses of all of the elements and their contributions to the forces on the whole structure, the resulting forces ΣR_g then equilibrate the forces R; thus

$$R = \Sigma R_g \tag{4}$$

Then, by substitution,

$$R = \Sigma a_g' T_g = \Sigma(a_g' k_g v_g) = [\Sigma(a_g' k_g a_g)]r \tag{5}$$

Defining

$$C = \Sigma a_g' k_g a_g \tag{6}$$

then

$$R = Cr \tag{7}$$

where C is the stiffness matrix of the structure and relates all of the displacements considered to the forces. Equation (7) therefore provides a complete solution for the structure acted upon by deformations and/or forces.

Method of solution

Partitioning the matrices of Eq. (7) gives

$$\begin{bmatrix} R_1 \\ R_0 \end{bmatrix} = \begin{bmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{bmatrix} \begin{bmatrix} r_0 \\ r_1 \end{bmatrix}$$

whence, with C_{01} equal to C_{10}',

$$R_1 = C_{00}r_0 + C_{01}r_1 \tag{8}$$

$$R_0 = C_{10}r_0 + C_{11}r_1 \tag{9}$$

Equation (9) gives the unknown displacements

$$r_1 = C_{11}^{-1}(R_0 - C_{10}r_0) \tag{10}$$

By substitution into Eq. (8), the unknown displacements are

$$p_1 = D_{00}P_0 + D_{01}D_{11}^{-1}(p_0 - D_{10}P_0) \quad (11)$$

Noting that

$$b = \{b_1 b_2 \ldots b_g \ldots\}$$
$$S = \{S_1 S_2 \ldots S_g \ldots\}$$
$$w = \{w_1 w_2 \ldots w_g \ldots\}$$
$$f = \begin{bmatrix} f_1 f_2 & \ldots f_g & \ldots \end{bmatrix}$$

then, from Eq. (1), considering all of the elements,

$$S = bP = b_0P_0 + b_1D_{11}^{-1}(p_0 - D_{10}P_0) \quad (12)$$

and, similarly,

$$w = fS = fb_0P_0 + fb_1D_{11}^{-1}(p_0 - D_{10}P_0) \quad (13)$$

As D equals $b'fb$, then

$$D_{00} = b_0'fb_0 = \Sigma b_{0g}'f_g b_{0g}$$
$$D_{01} = b_0'fb_1 = \Sigma b_{0g}'f_g b_{1g}$$
$$D_{10} = b_1'fb_0 = \Sigma b_{1g}'f_g b_{0g}$$
$$D_{11} = b_1'fb_1 = \Sigma b_{1g}'f_g b_{1g}$$

By substitution into Eq. (8), the unknown forces are

$$R_1 = C_{00}r_0 + C_{01}C_{11}^{-1}(R_0 - C_{10}r_0) \quad (11)$$

Noting that

$$a = \{a_1 a_2 \ldots a_g \ldots\}$$
$$v = \{v_1 v_2 \ldots v_g \ldots\}$$
$$T = \{T_1 T_2 \ldots T_g \ldots\}$$
$$k = \begin{bmatrix} k_1 k_2 & \ldots k_g & \ldots \end{bmatrix}$$

then, from Eq. (1), considering all of the elements,

$$v = ar = a_0r_0 + a_1C_{11}^{-1}(R_0 - C_{10}r_0) \quad (12)$$

and, similarly,

$$T = kv = ka_0r_0 + ka_1C_{11}^{-1}(R_0 - C_{10}r_0) \quad (13)$$

As C equals $a'ka$, then

$$C_{00} = a_0'ka_0 = \Sigma a_{0g}'k_g a_{0g}$$
$$C_{01} = a_0'ka_1 = \Sigma a_{0g}'k_g a_{1g}$$
$$C_{10} = a_1'ka_0 = \Sigma a_{1g}'k_g a_{0g}$$
$$C_{11} = a_1'ka_1 = \Sigma a_{1g}'k_g a_{1g}$$

It may be noticed that, for a structure under either applied forces or imposed displacements, Eqs. (10–13) will reduce to those that Argyris derived by a different approach.

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Hypersonic Sharp-Leading-Edge Problem for Axially Symmetric Bodies

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During the past several years, the so-called hypersonic sharp-leading-edge problem has been the subject of several investigations. All of these have been concerned with the particular case of a flat plate at zero incidence. The present note applies some of the methods developed for the flat plate to other bodies, in particular to pointed bodies of revolution at zero incidence. The present analysis is based on the viscous-layer model used by Oguchi[1] in his first analysis of the sharp-leading-edge problem. Essentially, Oguchi[1] assumes that 1) all rarefaction effects very near the tip are negligible; 2) the undisturbed flow is separated from the disturbed flow by a thin, nearly straight shock wave, which is attached at the leading edge and which satisfies the oblique shock relations; and 3) the flow behind the leading-edge shock wave forms a continuum viscous layer, which is sufficiently thin so that the Navier-Stokes equations can be truncated to their boundary-layer form.

Because of these simplifications, Oguchi's results show some inconsistencies[3, 4] (as do the results of Jain and Li[2] and Bendor[3] which also employ Oguchi's model). However, these analyses yield surface pressures very near the leading edge which are considerably more realistic than those predicted using strong interaction theory. Herein lies the justification for applying Oguchi's ideas to pointed-nose bodies of revolution.

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Analysis

Longitudinal curvature effects throughout the layer are assumed negligible. Furthermore, the gas is assumed to be perfect and to have constant c_p and Pr.† By the foremen-tioned assumption 3, the applicable equations are the boundary-layer equations that include transverse curvature terms [e.g., Eqs. (1.2–1.5, 1.8, and 1.9) of Ref. 5]. These equations are obtained from the full Navier-Stokes and energy equations by neglecting terms of the order of Δ/L, where Δ is of the order of the viscous-layer thickness, and L is of the order of the viscous-layer length. By assumption 2, the dependent variables u, v, p, ρ, and H must attain the values u_s, v_s, p_s, ρ_s, and H_s,‡ respectively, at some unknown but finite shock-wave distance $y_s(x)$. Furthermore, the values u_s, v_s, etc. are given by the oblique shock relations and hence depend on M_∞ and the local shock-wave angle ψ. Note that, neglecting longitudinal curvature, ψ is related to y_s by

$$dy_s/dx = \tan(\psi - \alpha) \quad (1)$$

For hypersonic conditions (i.e., $M_\infty \gg 1$), the oblique shock relations indicate that

$$v_s = u_\infty[(1 - \gamma)\sin\alpha + 2\sin(\psi - \alpha)\cos\psi]/(\gamma + 1) \quad (2)$$

† The coordinate system and notation used herein will agree with that of Yasuhara[5] except where otherwise defined.

‡ The subscript s will herein denote quantities evaluated directly behind the shock wave.